On the Notion of Continuity and Differentiability of Real Functions and Studying It with the Use of Computer

Sonia Barbosa Camargo Igliori
e-mail: sigliori@pucsp.br
Pontifícia Universidade Católica de São Paulo
São Paulo, São Paulo, BR

Celina Aparecida Almeida Pereira Abar
e-mail: abarcaap@pucsp.br
Pontifícia Universidade Católica de São Paulo
São Paulo, São Paulo, BR

Marcio Vieira de Almeida
e-mail: marcioalmeidasp@gmail.com
Pontifícia Universidade Católica de São Paulo
São Paulo, São Paulo, BR

Abstract
This paper aims at analyzing the use of computers when teaching differentiability and continuity in real-valued functions. The relation is approached in the case of a non-differentiable continuous function in a real interval. This example is found in an article written by David Tall and is used to evidence a way in which a computer helps the learning and teaching of concepts of Differential and Integral Calculus when didactic and meaningful materials are produced. Elements of Tall's theory on the advantages of the use of computers in Education, as well as the historical importance of the development of an example of a continuous non-differentiable function are presented in this paper. In addition, a case of a function defined as limit to a series of functions is explored. In addition, commands and tools, which are available in the software GeoGebra, are presented. As a result, we present tools, which will hopefully contribute to the practice as well as advancements in Mathematics Education at Higher Education level.

1. Introduction

This paper aims at analyzing the use of computers when teaching differentiability and continuity in real-valued functions, and more than that, the relation between these two concepts. In order to do this, we explore an example, presented by Tall, with the aid of software, which we hoped to indicate the advantages of the use of the computer when teaching Differential and Integral Calculus.

We have developed some elements of Tall's theory on the use of computers in teaching and we have highlighted the historical importance to the development of Mathematics with the building an example of a continuous non-differentiable function. The example we dealt with is a function defined as limit in a series of functions. We have indicated the commands and tools available in the GeoGebra software, which are used to build this function, which was named, by the English researcher, the "blancmange function" because of its graphical representation of partial sums of a series.

The choice of this example is made because it is contrary to the student's intuition and reinforces the idea that there is a real-valued function, which is continuous, but it is not necessarily differentiable. We have also chosen this example because there is an adequate software available to build it.

The potentiality of the use of computers when teaching advanced Mathematical topics is mentioned by Tall [7]:

> […] to use the computer to visualise mathematical concepts in helpful ways in calculus and analysis. Imaginative use of graph-plotters and graphic calculators has enabled students to cope more meaningfully with concepts such as differentiation through the notion of "local straightness", integration through area summation, and solving (first order) differential equations by visually building up solution curves with given gradient. During this time, I became increasingly aware of the limited concept imagery afforded by graph-plotters that only draw reasonably smooth graphs given by formulae (p. 2).

According to Tall, the computer, when containing adequate software, may be used to provide images, which will help in the development of Calculus and Analysis [1].

By using the example of blancmange function, it is possible "to give a refined conceptual explanation of continuity and differentiation, which are formally correct and have a suitable pictorial interpretation" [5].

In Tall's formulations, the graphical representation of the differential function, when enlarged to a determined portion, looks like, locally, a segment of a straight line. Afterwards, the researcher formulated the notion of cognitive roots "local straightness", which is based on the perception that tiny part of certain graph under high magnification eventually looks virtually straight [6]. This notion would be appropriate to the development of the concept of derivative because "it allows the gradient function to be seen as the changing gradient of the graph itself" [8].

By the notion of local straightness, it would be possible to stimulate the student's imagination to conceive how a graphic representation of a continuous and non-differentiable function at the points of the domain would be. A characteristic of this representation would be the following: it should keep the "beak", not mattering how much this function is enlarged. The blancmange function would be an example of this fact, because of the way it is defined.

Another reason related to the example presented in this paper is that, in some historical episodes concerning the constitution of the formal concept of continuity, mathematicians, from the beginning of the XIX century, conjectured that continuous functions had points in which they were differentiable. This way, an example of a continuous non-differentiable function in all points of its domain would contradict such conjecture. Besides, as Araújo e Fávaro [2] describe

> […] a lot of mathematicians believed that continuous functions had derivatives with a "meaningful" number of points and some mathematicians tried to give theoretical justifications for this fact, such as A. M. Ámpere, in a paper published in 1806. However, until the beginning of the 19th Century, the main concepts of Calculus did not have adequate logical justification and Ámpere's paper failed to explain, due to the limitations of the definitions back then. In 1872, K. Weierstrass published a paper that "shocked" the mathematics community proving that this conjecture was false. More precisely, he built and example of a continuous function which was not differentiable at any point. (p. 5, translated by the authors).

Concerning the concepts of continuity and differentiability of a real function, the following result is stated: Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in X$, if f is differentiable in x_0 then f is continuous in x_0.

The reciprocity of this result is false, because there are continuous functions in a determined point of the domain, which are not differentiable at this point. In general, the counterexample to the reciprocity of the theorem is the absolute value function, that is, the real function defined by $f(x) = |x|$, in $x = 0$, it is a continuous function in 0, but not differentiable in 0, because $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist. However, the differentiability is not guaranteed only at the zero point in a function. On the other hand, the blancmange function is a continuous function in all points of the domain, but not differentiable in any of those points. Besides it is difficult to conceive it and it is not commonly presented.

In order to build such example we have chosen the software, written in Java, of Dynamic Geometry, GeoGebra, because it is free, has simple and intuitive interface and enables the development of activities which gather Geometry, Algebra and Calculus. It is possible to elaborate and modify applets, either to use in the classroom or to make them available on internet websites. This software holds all tools and commands, which enable the building of the aforementioned function, once it is the limit to a series of functions. This way, we intend to show that GeoGebra could be used in the introduction the concepts of both Differential and Integral Calculus and the Real Analysis.

In the section 2, we present the "blancmange function". In section 3, we build the first term of the series of function, in which the limits is blancmange function with GeoGebra software. In the section 4, we show the software tools, which is used in the building of terms of sequence and series of functions. Finally, we present the considerations related to the graphic representation of the function and indications of the way in which the software GeoGebra may be used in teaching and learning mathematical concepts approached in Higher Education.

2. The definition of the real "blancmange function"

Let $f_n : [0,1] \to \mathbb{R}$, which general term of this sequence is $f_n : [0,1] \to \mathbb{R}$, with $f_n(x) = \dfrac{1}{2^{n-1}} f(2^{n-1} \cdot x)$ and $f \colon \mathbb{R} \to \mathbb{R}$, is defined by the following sentence $f(x) = d(x, \mathbb{Z})$[1]. We define the "blancmange function" as the function $b \colon [0, 1] \to \mathbb{R}$, with

$$b(x) = \sum_{i=1}^{\infty} f_i(x)$$

The function b satisfies:

It is a continuous function:

It is fact that $|f(x) - f(y)| \le |x - y|, \forall x, y \in \mathbb{R}$, then f is a continuous functions.

It is fact that $f_i = f \circ g_i$ where $g_i \colon \mathbb{R} \to \mathbb{R}$, $g_i(x) = 2^{i-1} \cdot x$. As g_i and f are continuous functions then f_i is a continuous function $\forall i \in \mathbb{N}$.

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[1] In the metric space $(\mathbb{R}, |.|)$, the function $d \colon \mathbb{R} \to \mathbb{R}$, $d(x, \mathbb{Z}) = inf \left\{ |x - z|, \text{ for } z \text{ an integer} \right\}$ is the function that associate each x to the distance between x and the set $\mathbb{Z}$.

It is fact that the numerical series $\left(\sum_{n=1}^{m}\left(\frac{1}{2}\right)^{n-1}\right)_{m \in N}$ converges. As $|f_i(x)| \leq \left(\frac{1}{2}\right)^{i-1} \forall i \in \mathbb{N}$, we

conclude that the series of functions $\left(\sum_{i=1}^{n} f_i\right)_{n \in N}$ converges uniformly to b. The uniform convergence

of the series of function guaranteed the continuity of b.

It is a nowhere differentiable function:
To show that the function f is nowhere differentiable, we consider the following result:
Let h: $[a, b] \rightarrow \mathbb{R}$, (u_n) and (v_n), with $u_n \in [a,b] \forall n \in \mathbb{N}$ and $v_n \in [a,b] \forall n \in \mathbb{N}$, numerical sequence such as $u_n \leq x \leq v_n$ (with $u_n < v_n$) for all n in $\mathbb{N}$ and $u_n - v_n \rightarrow 0$. If h is a continuous function and $h'(x)$ exist, so

$$\frac{h(v_n) - h(u_n)}{v_n - u_n} \rightarrow h'(x)$$

To show that the function b is a nowhere differentiable function, we will define two sequences that contradict the last result announced.
Let $x \in [0,1]$ a fixed number, b: $[0, 1] \rightarrow \mathbb{R}$ the blancmange function and, by contradiction, we suppose that $b'(x)$ exists.
Let $D = \left\{i \cdot 2^{-m} \mid i, m \text{ an integer}\right\} \cap [0,1]$ and for each number $i \cdot 2^{-m}$, for m and i integer, $2^k(i \cdot 2^{-m}) = i \cdot 2^{k-m}$ and $i \cdot 2^{k-m} \in \mathbb{Z}$ for all $k \geq m$. If f: $\mathbb{R} \rightarrow \mathbb{R}$, is defined by $f(x) = d(x, \mathbb{Z})$ and p an integer then $f(p) = 0$.
Let u_m, v_m in D, where $u_m = (i-1) \cdot 2^{-m}$, $v_m = i \cdot 2^{-m}$ and i is an integer that guarantee the following relation $u_m \leq x < v_m$. In this case, $v_m - u_m = 2^{-m}$ and $v_m - u_m \rightarrow 0$.
It is a fact that

$$b(u_k) = \sum_{k=1}^{m} \frac{1}{2^{k-1}} \cdot f_k(u_k)$$

If $x \in \left[2^k u_m, 2^k v_m\right]$ then $x \in \left[\frac{i-1}{2^{k-m}}, \frac{i}{2^{k-m}}\right]$, so for $0 \leq k < m$

$$\frac{f_m(v_m) - f_m(u_m)}{v_m - u_m} = \frac{1}{2^k} \cdot \frac{f(2^k \cdot v_m) - f(2^k \cdot u_m)}{v_m - u_m} = \frac{\pm 2^{k-m}}{2^{k-m}} = \pm 1$$

In reason of that,

$$b'(x) = \frac{b(v_n) - b(u_n)}{v_n - u_n} = \sum_{k=1}^{m} \pm 1$$

When $m \rightarrow \infty$, the series does not converge. This fact contradicts the assumption that $b'(x)$ exists. Since $x \in [0,1]$ is arbitrary, we can conclude the function b is nowhere differentiable. In the next section, we present the first term, commands and tools available in the GeoGebra

software, which may be used to the building of a graphic representation of the partial sums of the series $\left(\sum_{i=1}^{n} f_i \right)_{n \in N}$, in which the limit is the "blancmange function".

3. Building of the "sawtooth function"

In order to build the "blancmange function" it is necessary, at first, to build a real function, which can be called "sawtooth function". This function is an example of a continuous function, which is non-differentiable at an infinite number of points. This construction is necessary, because the first term of a series of functions, in which the limit is the "blancmange function", is the "saw tooth function", restricted to the interval [0,1]. The graphic representation of this function is:
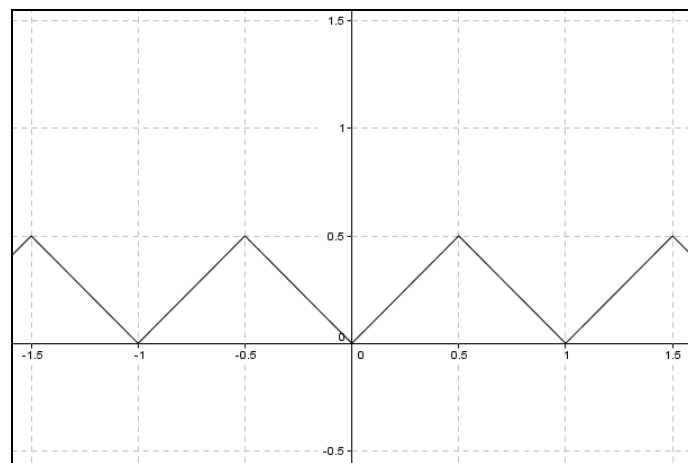
Figure 1: *the graphic representation of the "sawtooth function"*

The function f_1: [0, 1] → ℝ, is defined by the following sentence $f(x) = d(x, \mathbb{Z})$. The software GeoGebra do not have a function that describes this function. We need do some considerations to build this function with other sentence. In this case, $f_1 : [0,1] \to \mathbb{R}$, is defined by the following sentence $f_1(x) = \left| x - \{x\} \right|$, and $\{x\}$ denotes the image of the real function { }. In order to define the function { }, we consider the fact that if x is a real number, then $z \in \mathbb{Z}$ exist, and $d \in [0,1)$ so that $x = z + d$. So:

$$\{x\} = \{z + d\} = \begin{cases} z & \text{if } 0 \le d < 0,5 \\ z+1 & \text{if } 0,5 \le d < 1 \end{cases}$$

Note that to all $x_0 \in [0,1)$, so $x_0 = 0 + d$. Considering this, if $0 \le d < 0,5$ so $\{x_0\} = 0$, and if $0,5 \le d < 1$ so $\{x_0\} = 1$. This way, to $0 \le d < 0,5$, $f_1(x_0) = \left| x_0 - \{x_0\} \right| = \left| d - 0 \right| = d$, and, to $0,5 \le d < 1$, $f_1(x_0) = \left| x_0 - \{x_0\} \right| = \left| d - 1 \right| = 1 - d$. Then, it is valid that $0 \le f_1(x) \le 0,5$, to $x \in [0,1)$. If $x_0 = 1$, it is valid that $f_1(1) = \left| 1 - \{1\} \right| = 0$. So, $0 \le f_1(x) \le 0,5$, to $x \in [0,1]$.

In order to build this function by using the GeoGebra software, pre-defined functions are necessary. Those are explained in the next paragraphs.

Firstly, it is necessary to use the pre-defined function *round()*. According to the software manual [3], this command is described as round and it does the following operation: associates a real number x to the whole number, which is closest to x. When typing, in the Input Bar, the command *round(x)*, the graph of the real function, given by $g(x) = \{x\}$, is sketched. The graphic representation of g function will appear in the Graphics view.
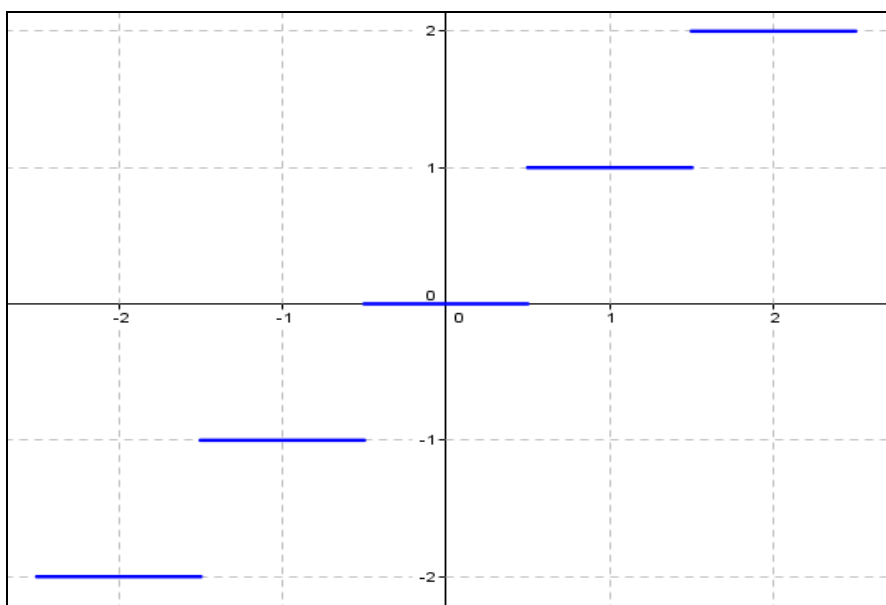
Figure 2: *the graphic representation of the g function, given by the sentence g(x) = {x}.*

Proceeding the building of the "sawtooth function", it is necessary to build an f, function, in which the domain and image are real number sets, with the following sentence $f(x) = |x - \{x\}|$. In order to build it, it is necessary to use, besides the pre-defined *round()* function, the *abs()* function, called absolute value, which brings the same definition of the absolute value in a real number.

When using the pre-definite functions, all you have to do is type, in the Input Bar, the following: "*f(x) = abs(x − round(x))*". As a result, the graphic representation of the "sawtooth function" will appear in the Graphics view, as shown in Figure 1.

In order to build the first term of sequence of f_n function, it is necessary to limit the domain of the f function to the interval [0,1]. It is possible to do it in GeoGebra, with the boolean command "If". Following the software manual [3], the command brings the following structure: "If[<Condition>, <Then>]", and it can be used when building a function in which the domain is a subset of real numbers. With the objective of exemplifying the use of this command, consider the function $h : [-2,2] \to \mathbb{R}$, given by the following sentence $h(x) = \dfrac{3}{2}x - 1$. In order to sketch the graphic of this function, it is necessary to type the following commands in the Input Bar, "$h(x) = \text{If} \left[-2 \leq x \leq 2, \dfrac{3}{2}x - 1\right]$".

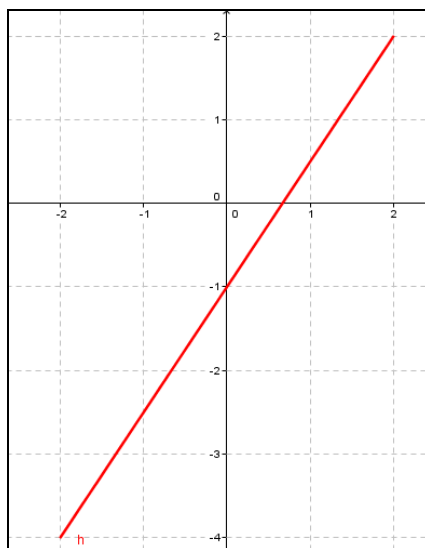In Figure 3, find the graphic representation of the *h* function, in the Graphics view:

Figure 3: *the graphic representation of the function h:* [-2,2] → ℝ, *given by the sentence* $h(x) = \frac{3}{2}x - 1$, *in the Graphics view.*

To indicate the alteration of the domain of the function, the following sentence is exhibited in the Algebra view (Figure 4):
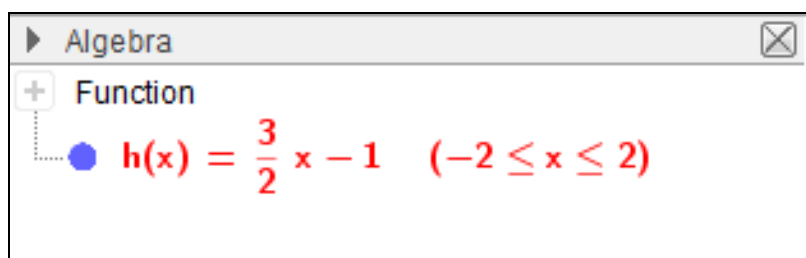

Figure 4: *the function representation h:* [–2, 2] → ℝ, *given by the sentence* $h(x) = \frac{3}{2}x - 1$, *in the Algebra view.*

Returning to the construction of the first term of the sequence of the f_n functions, it is necessary to type the following in the Input Bar:

"f_1(x) = If[0 ≤ x ≤ 1, abs(x – round(x))]"

In Figure 5, it is possible to see the graphic representation of the first term of the sequence of functions, in which the limit is the same as the "blancmange function":
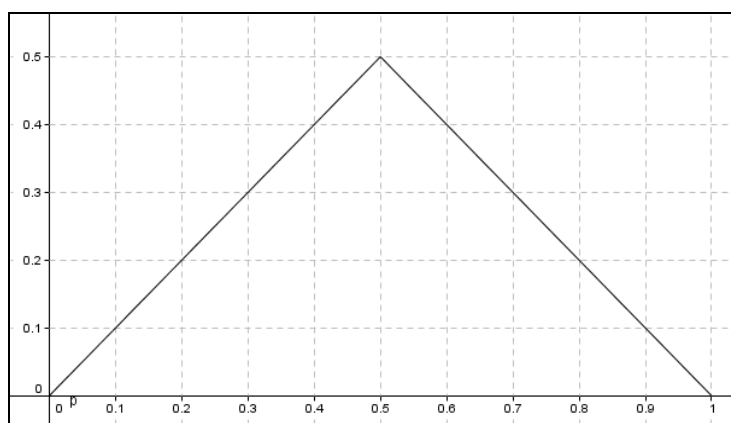
Figure 5: *the representation of the function f_1:* [0,1] $\rightarrow \mathbb{R}$*, given by the sentence* $f_1(x) = |x - \{x\}|$.

In next session, we will present other tools of the software, which are used to build the general term of the sequence of functions and the series of functions.

4. The construction of the "blancmange function".

In such building, other tools available in GeoGebra are used. These tools are appropriate to explore sequence of functions commonly presented in a course of Analysis. Initially, it is shown how it is possible to represent the terms of that sequence.

We may represent a sequence of functions in GeoGebra with the command "Sequence". According to the software manual ([3] see pp. 60 − 61) it is necessary to type in the Input Bar: "Sequence[<Expression>, <Variable>, <Start Value>, <End Value>]". Such command provides the list of objects created using the "Expression" given, according to the "Variable", determined by the user and that must be part of the typed "Expression", that ranges from the "Start Value" to the "End Value".

For example, to build the terms of the sequence $(g_i)_{i \in N}$, $g_i : [0,1] \rightarrow \mathbb{R}$, defined by $g_i(x) = ix(1-x)^i$, it is necessary to type the following commands, in the Input Bar:

"Sequence[If[$0 \le x \le 1, i^* x^* (1-x)$^$i$] , i, 1, 30]"

As a result, the thirty terms of the sequence of functions are represented in the Graphics view (Figure 6):
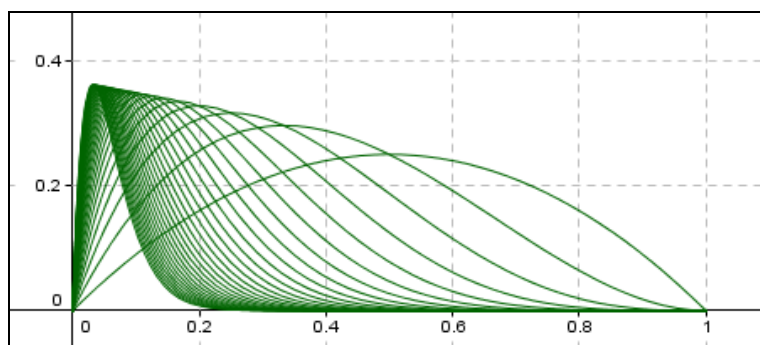

Figure 6: *the representation of the thirty terms of the function* g_i: [0,1] $\rightarrow \mathbb{R}$, $g_i(x) = ix(1-x)^i$.

With the help of the representation of terms of the functions, it is possible to infer that sequence converges punctually to the function identically zero.

Another way to implement a sequence of functions is using the slider tool, available in GeoGebra. According to the software manual [3], a slider is the graphic representation of a parameter, which is possible to be a number or an angle, and when it modified produces determined alteration in the element associated to the parameter. This tool may be used to build a representation of the terms of a sequence. Consider a sequence of functions built previously.

Now, when clicking in the slider tool, which the icon is [icon], a dialog box will open with some options. Firstly, a name is given to the "Slider". In our case the name given is "n". The option "Integer" is selected and in the icon "Interval", the maximum and minimum values of the parameters must appear. For instance, the minimum value attributed is 1 and the maximum 30 (Figure 7).
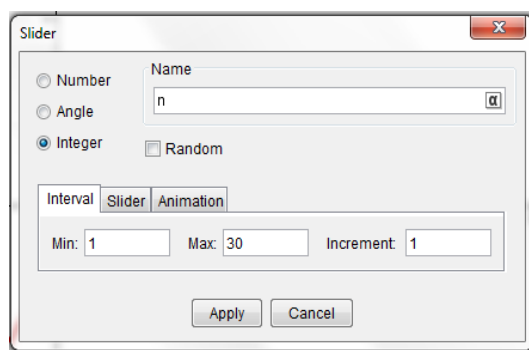

Figure 7: *the dialog box used to build a "Slider".*

When the "Slider" is ready, it should be typed in the Input Bar:
"Sequence[If[$0 \leq x \leq 1$, $i^{*}x^{*}(1-x)$^i] , i, 1, n]"

The n terms in the sequence of functions $(g_n)_{n \in N}$ are exhibited when we move the slider control in the "Graphics" in GeoGebra.

In order to build the series of functions in which the limit is the "blancmange function", all the objects and functions built previously are used. Them the following command should be typed in the Input Bar:
"Sequence[If[$0 \leq x \leq 1$,(f(2^(j - 1)*x)) / (2^(j - 1))], j, 1, n]"

The command "if" is used to determine: the domain of the functions which compose the sequence, the interval [0,1], (f(2^(j - 1)*x)) / (2^(j - 1)) represents the function $f_i : [0,1] \rightarrow \mathbb{R}$, given by the sentence $f_i(x) = \dfrac{1}{2^{i-1}} f(2^{i-1} \cdot x)$, being f the real function given by the following sentence $f(x) = \left| x - \{x\} \right|^2$. In this part, the typed command is in function of j, because this variable is necessary to the command "Sequence". The numeral 1 is the minimum value of the sequence and the maximum value is n, which is the "slide control", created previously. The ten terms of the sequence of functions $(f_n)_{n \in N}$ are represented in Figure 8:

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[2] Note that this function is built with the inclusion of the following commands in the Input Bar: "$f(x) = abs(x - round(x))$". Observe that, for the command to make sense, it is necessary that this function be built in the same file.
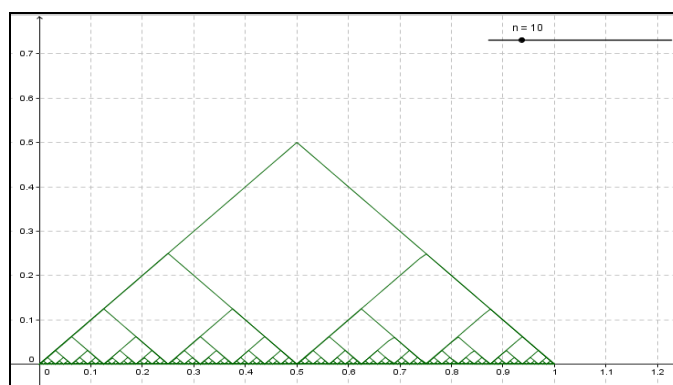
Figure 8: *the ten terms of the sequence* $(f_n)_{n \in N}$.

The built sequence is named *list*1. A list, to GeoGebra, is a group of objects created by the user. In this case, the *list*1 is a list with the *n* terms of the sequence of functions. The software has commands, which can order, compare, add elements and operate with elements belonging to a list (see for example [3] page 58). Our objective is to build a graphic representation of the partial sum of the series of functions,

$$b_n(x) = \sum_{i=1}^{n} f_i(x)$$

In order to build b_n you need to use the command "Sum[<list>]", because it sums all the elements of a certain list. This way, you only need to type, in the Input Bar, the following: "Sum[*list*1]"

As a result, in the Graphics view, the partial sum in the series of function is exhibited. The partial sum $\sum_{i=1}^{30} f_i(x)$, as $(f_n)_{n \in N}$ is the sequence of built functions in this session, and is

represented in Figure 9.

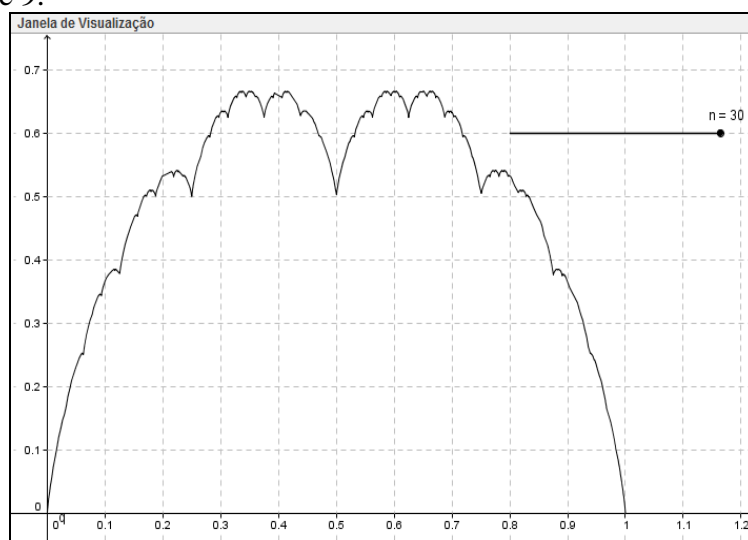Figure 9: *the representation of the partial sum* $\sum_{i=1}^{30} f_i(x)$, *as* $(f_n)_{n \in N}$ *is the sequence of*

functions built in this article.

5. Final remarks

In this paper, we have exposed a representation of a continuous function at all points of the domain, and non-differentiable at any. This function was approached in Tall (for more details see [7] and [8]).

According to Tall [7], one of the objectives of using this example and the introduction of notion of cognitive root of local straightness is the possibility of stimulating the imagination of the student to conceive how the graphic representation of a continuous non-differentiable function at the points of the domain would be. In this perspective, the graphic representation of this function should remain with 'beaks'', not mattering how much this function is enlarged.

However, it is evident that the representation, itself, does not guarantee that the "blancmange function" is continuous everywhere and differentiable nowhere. If the reader is interested in verifying other reference that explains the reason why the presented function is non-differentiable everywhere, we suggest the following reference Tall [5]. It is possible to find another example of a continuous non-differentiable function, developed by the Mathematician B. van der Waerden, in 1930 (see [2]) and other examples of functions continuous and non-differentiable everywhere function see [9], [11] and [12].

In Lima [4], we found an indication that the set of continuous functions $f : I \to \mathbb{R}$, which do not have derivative at any point of the interval $I,$ "in a natural sense, brings 'the majority' of continuous functions $f : I \to \mathbb{R}$" (p. 195, translated by the authors), which seems to be a very surprising fact.

Another point, it is to exhibit tools, commands and predefined functions that are available in the GeoGebra software. Those enable the teacher to elaborate meaningful didactic materials, which can be used to teaching and learning the concepts approached in Higher Education, mainly in the teaching of Differential and Integral Calculus and Real Analysis.

Finally, it is expected that both the built example and the presented tools might help in future field researches and foment discussions that will contribute to the advance of Mathematics Education at Higher Education.

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